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INSTABILITY OF SOLITARY WAVES IN NON-LINEAR COMPOSITE MEDIA†

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A problem on the dynamic instability of soliton solutions (solitary waves) of Hamilton's equations, describing plane waves in non-linear elastic composite media with or without anisotropy, is considered. In the anisotropic case, there are two two-parameter families of solitary waves: fast and slow and, when there is no anisotropy, there is one three-parameter family. A classification of the instability of solitary waves of the fast family in the anisotropic case and of representatives of families of solitary waves, the velocities of which lie outside of the range of stability when there is anisotropy and when there is no anisotropy, is given on the basis of a numerical solution of a Cauchy problem for the model equations. In this paper, the fundamental equations describing plane waves in non-linear, anisotropic, elastic composites are derived, the Hamilton form of the basic equations is presented, the symmetries in the anisotropic and isotropic cases are considered, the conserved quantities and the soliton solutions, that is, the solitary waves are presented, the nature of the instability of representatives of all three families is investigated, brief formulation of the results is presented and problems of the instability of the fast family in the anisotropic case and of representatives of the families, the velocities of which lie outside of the range of stability in the presence and absence of anisotropy (explosive instability), are discussed. © 2002 Elsevier Science Ltd. All right reserved.

It is well known that the properties of a composite material, which is described by averaged equations, (see [1, 2], for example) and the properties of its constituent components are substantially different. In particular, the case of the initiation of dispersion is typical in spite of the fact that, in each of the elastic materials constituting the composite material, there is no dispersion [1]. A composite material consisting of elastic materials with a non-linear equation of state is therefore a dispersing medium in which waves can propagate, which are the result of the interaction of non-linear and dispersion effects, including solitary waves. The question of the possibility of observing solitary waves in practice is naturally associated with the dynamic stability of these waves.

The dynamic stability of solitary waves in a non-linearly elastic composite material has been investigated [3, 4] both when there is anisotropy and no anisotropy, and the sufficient conditions for the non-linear stability of the different families of solitary waves which branch from a rest state have been determined. The hypothesis has also been put forward that, when these conditions are violated, solitary waves are unstable. The correctness of these hypotheses is established below by means of numerical calculations and a classification of the form of the instability of solitary waves is presented.

1. FORMULATION OF THE PROBLEM, SYMMETRIES AND SOLITON SOLUTIONS

Plane wave motions in an inhomogeneous, non-linear, elastic medium (composite) are investigated, when the displacements w_α, the strains $u_\alpha = \partial w_\alpha/\partial x$ and the velocities of the particles v_α ($\alpha = 1, 2, 3$) depend on a single spatial variable, the Cartesian coordinate $x = x_3$ and the time t. We shall consider incompressible elastic media when u_3 and v_3 are constant. These constants can be equated to zero without any loss in generality.

Despite the fact that the motions of a non-linear elastic body are described by a hyperbolic system of equations [5], the existence of an internal inhomogeneous structure in a material at the macrolevel leads to wave dispersion [1, 2]. It is well known [2] that the dispersion terms can be introduced, when averaging, into equations of the form

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$$\mathfrak{Q}\mathbf{v} = -\rho(y_1, y_2, y_3)\partial_{tt}\mathbf{v} + \mathfrak{Q}^0\mathbf{v}, \quad \mathfrak{Q}^0\mathbf{v} = \sum_{i,j=1}^{3} \frac{\partial}{\partial x_i}\left(A_{ij}(y, y_2, y_3)\frac{\partial \mathbf{v}}{\partial x_j}\right), \quad y_j = \frac{x_j}{\varepsilon}$$

by two methods. Here, ρ and $A_{ij} = A_{ij}^T$ are periodic functions with a period of unity, $\varepsilon \ll 1$ which means that the period of the inhomogeneity of the medium is much less than the characteristic length of the waves being studied. In the case when $A_{ij} = A_{ij}(y)$, $y = x_1/\varepsilon$ and a wave propagates in the x_1 direction, the dispersion terms of lower order which arise during averaging have the form (if $\mathbf{v}$ is a two-dimensional vector) $\mathbf{v}''' \times \mathbf{b}$, where $\mathbf{b}$ is a constant pseudovector and the prime denotes a derivative with respect to the spatial variable. In the case when the direction of propagation of the wave is orthogonal to the direction of periodicity of the medium, the additional dispersion terms in the lower order with respect to ε have the form $\mathcal{M}\mathbf{v}'''$, where $\mathcal{M}$ is a symmetric matrix. We will further assume that there is no wave anisotropy and that $\mathcal{M} = \operatorname{diag}\{m, m\}$. As regards the elastic medium, we will assume that the non-linearity, anisotropy and dispersion are small and are represented by first-order terms. The system of basic equations can then be written in the form [6]

$$\partial_t u_i - \partial_x v_i = 0, \quad \rho_0\partial_t v_i - \partial_x(\partial\Phi/\partial u_i) + m\partial_{xxx}u_i = 0, \quad i = 1, 2 \tag{1.1}$$

Here ρ_0 is the mean density of the material and Φ is the elastic potential, which is given by the expression

$$\Phi = \frac{1}{2}f(u_1^2 + u_2^2) + \frac{1}{2}g(u_2^2 - u_1^2) - \frac{1}{4}\varkappa(u_1^2 + u_2^2)^2$$

The constants $g > 0$ and $\varkappa$ characterize the anisotropy and non-linearity, respectively. Expressions for the constants f, g and $\varkappa$ are given in [5]. A dispersion term with $m > 0$ appears in the equations of motion (the second pair of equations in (1.1)), for example, in the case when a homogeneous, elastic, easily deformed medium contains homogeneously distributed rods which have a sufficient flexural rigidity and are arranged parallel to the x axis [6] A cubic non-linearity in the isotropic elasticity has been considered earlier in [7] in a non-linear problem on transverse vibrations which are excited in an infinite elastic layer by the periodic action of an external tangential force on one of the plane boundaries.

Equations (1.1) can be written in the Hamilton form

$$\partial_t \mathbf{w} = \mathcal{T}\frac{\delta E(\mathbf{w})}{\delta \mathbf{w}}, \quad \mathbf{w} = \begin{Vmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{Vmatrix}, \quad \mathcal{T} = \begin{Vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{Vmatrix}\partial_x, \quad \frac{\delta}{\delta \mathbf{w}} = \begin{Vmatrix} \delta/\delta u_1 \\ \delta/\delta u_2 \\ \delta/\delta v_1 \\ \delta/\delta v_2 \end{Vmatrix} \tag{1.2}$$

$$E = \frac{1}{2}\int_{-\infty}^{\infty} [v_1^2 + v_2^2 + \mu_1 u_1^2 + \mu_2 u_2^2 - \frac{\varkappa}{2\rho_0}(u_1^2 + u_2^2)^2 + \frac{m}{\rho_0}(\partial_x u_1)^2 + \frac{m}{\rho_0}(\partial_x u_2)^2]dx$$

$$\mu_1 = (f - g)/\rho_0, \quad \mu_2 = (f + g)/\rho_0$$

The Hamiltonian E is obviously constant by virtue of system (1.2). Moreover, it is easy to see that the functional

$$Q = \int_{-\infty}^{\infty} [u_1 v_1 + u_2 v_2]dx$$

is also invariant. The vector functional

$$\mathbf{A} = \int_{-\infty}^{\infty} \mathbf{w}dx$$

is also a formally conserved quantity. Equations (1.1) and (1.2) in the anisotropic case ($g \neq 0$) and the functionals E, Q and $\mathbf{A}$ are invariant under the group of translations

$$T(s)\mathbf{w} = \mathbf{w}(x + s) = \exp(s\partial_x)\mathbf{w}(x), \quad s \in \mathbf{R}$$

The functional Q is a conserved quantity as a consequence of the translational invariance of (1.1).

The system of equations (1.2) has additional conserved quantities in the special case of degeneracy of the anisotropy ($g = 0$). In this case, we have additional rotational symmetry

$$G(\varphi)\mathbf{w} = \exp(\mathcal{A}\varphi)\mathbf{w}, \quad \varphi \in \mathbf{S}^1; \quad \mathcal{A} = \operatorname{diag}\left(\begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}, \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}\right)$$

where $\mathbf{S}^1$ is the periphery. As a result of the existence of rotational symmetry, the quantity

$$U = \frac{1}{2}\int_{-\infty}^{\infty} [y_1 v_2 - y_2 v_1 + u_2 w_1 - u_1 w_2]dx; \quad \partial_x y_i = u_i, \quad \partial_x w_i = v_i$$

is formally conserved. The Hamiltonian E and the functional Q are also invariant under the group of rotations.

Soliton solutions of system of equation (1.1) are travelling solitary waves which rapidly decay at infinity. After substituting $\mathbf{w} = \mathbf{w}(\xi)$ into (1.1), where $\xi = x - Vt$ and V is the constant velocity of propagation of the wave, and a single integration using the conditions of decay at infinity, we obtain

$$v_i = -Vu_i, \quad \frac{m}{\rho_0}\ddot{u}_i = (\mu_i - V^2)u_i - \frac{\varkappa}{\rho_0}u_i(u_1^2 + u_2^2) \tag{1.3}$$

The dots denote differentiation with respect to the variable ξ.

Equations (1.3) for the solitary waves can be written in the equivalent form

$$\frac{\delta E(\phi_V)}{\delta \mathbf{w}} + V\frac{\delta Q(\phi_V)}{\delta \mathbf{w}} = 0; \quad \phi_V = \{u_1^s, u_2^s, v_1^s, v_2^s\}^T, \quad v_i^s = -Vu_i^s \tag{1.4}$$

When $u_i > 0$, $\varkappa > 0$, Eqs (1.3) have soliton solutions (which are denoted by the superscript s in (1.4)), to describe which it is convenient to introduce the function

$$S(\xi, \nu) = \pm\frac{\sqrt{2\rho_0\varkappa^{-1}(\nu - V^2)}}{\operatorname{ch}\sqrt{\rho_0 m^{-1}(\nu - V^2)}\xi}$$

In the anisotropic case, Eqs (1.3) have two families of soliton solutions which branch from a state of rest

$$1)\ u_1^s = S(\xi, \mu_1), \quad u_2^s = 0; \quad 2)\ u_1^s = 0, \quad u_2^s = S(\xi, \mu_2) \tag{1.5}$$
$$V \in I, \quad I = \{V, V^2 < \mu_{1,2}\}$$

Each of the families (1.5) is a two-parameter family: the velocity V and the displacement s along the ξ coordinate serve as the parameters. We shall henceforth call the first family of solitary waves in (1.5) the slow family, as this family exists for the smaller range of speeds, and the second, the fast family.

When there is no anisotropy $\mu_1 = \mu_2 = \mu$, there is additional rotational symmetry: if $\phi_V = \{u_1^s, u_2^s, v_1^s, v_2^s\}^T$ is a soliton solution of Eqs (1.3) then $\exp(\mathcal{A}\varphi)\phi_V$, $\varphi \in \mathbf{S}^1$ will also be a soliton solution. It is therefore sufficient to consider a single specific case with a fixed φ:

$$u_1^s = S(\xi, \mu), \quad u_2^s = 0; \quad V \in I, \quad I = \{V, V^2 < \mu\} \tag{1.6}$$

The family of solitary waves, of which (1.6) is a representative, is a three-parameter family with the angle of rotation φ serving as the third parameter.

The stability of the boundary states of the functional E (of the soliton solutions) results from the possibility of constructing a functional (which has the sense of a Lyapunov function) possessing a local minimum in the neighbourhood of the soliton solution ϕ_V. In the case of infinite dimensional Hamilton systems of the form of (1.2),

$$R(\mathbf{w}) = E(\mathbf{w}) + VQ(\mathbf{w})$$

is used as such a functional [8]. By virtue of the symmetry properties of the translationally invariant equations, the functional $R(\mathbf{w})$, as a rule, does not have a local minimum at the point ϕ_V in the whole of the functional space X of the solutions. A minimum can be reached in the non-linear subset of the space X, which is defined as follows [8]:

$$M = \{\mathbf{w} \in X, Q(\mathbf{w}) = Q(\phi_V)\}$$

The existence of a local minimum of the functional $R(\mathbf{w})$ at the point $\phi_V \in M$ in M is sufficient for the non-linear dynamic stability of the soliton ϕ_V.

By virtue of (1.4), the behaviour of the functions $R(\mathbf{w})$ in the neighbourhood $\mathbf{w} = \phi_V$ is completely defined by the spectral properties of the self-adjoint operator

$$\mathcal{H} = \frac{\delta^2 E(\phi_V)}{\delta \mathbf{w}^2} + V\frac{\delta^2 Q(\phi_V)}{\delta \mathbf{w}^2}$$

When $R(\mathbf{w})$ contracts into the submanifold M, the existence of a single unstable direction of the operator $\mathcal{H}$ is permitted, that is, this operator can have a single, simple negative eigenvalue for the dynamic stability of ϕ_V to exist. In this case, the positive spectrum of the operator $\mathcal{H}$ must be separate from zero. When these spectral properties are satisfied, the functional $R(\mathbf{w})$ has a local minimum at the point ϕ_V in M (which implies the stability of ϕ_V) subject to the condition that $\partial Q(\phi_V)/\partial V > 0$ [8]. It has been shown [3, 4] that the sufficient conditions for stability indicated above are satisfied when $V \in I_{st}$ for the slow family of solitary waves $I_{st} = \{V \in I, V^2 > \mu_1/2\}$ and for the isotropic solitary waves $I_{st} = \{V \in I, V^2 > \mu/2\}$ (stability ranges), that is, there is dynamic stability of the solitary waves belonging to the indicated families over the stability range. It has been postulated that solitary waves belonging to the above mentioned families but outside the stability range I_{st}, as well as all the solitary waves belonging to the fast family, are unstable [3, 4]. This hypothesis has been confirmed by a numerical analysis of the solutions of the Cauchy problem for Eqs (1.1).

Numerical results of the investigation of the nature of the instability, outside the stability range, of solitary waves belonging to the slow and isotropic families (large amplitude slow and isotropic waves), as well as of solitary waves belonging to the fast family, are presented later.

2. EXPLOSIVE AND DECAY INSTABILITY

In this section, the evolution of initial data of the solitary-wave type (1.5), (1.6) is investigated numerically using a three-layer scheme with central differences. This scheme has already demonstrated its effectiveness in solving a number of problems on the modelling of solitary waves and discontinuities [9–11]. It ensures the conservation, in the numerical solution, of the main properties of the model, that is, its conservative character and invertibility. Also, in this case, the use of this scheme has fully confirmed the proposals, which are expected from theoretical arguments, concerning the instability of solitary waves in certain ranges of the velocity parameter.

Calculations were carried out for all possible typical combinations of the parameters: fast or slow wave, the isotropic or anisotropic case, in the presence or absence of rotation of the initial data in the isotropic case (that is, whether a second deformation component u_2 occurs as the result of an orthogonal transformation applied to (1.6)), and whether the velocity lies inside or outside the stability range. The numerical calculations fully confirmed the theoretical results [3, 4] concerning stability. Explosive instability is observed outside the stability ranges and, also, when $V^2 < \mu_2/2$ in the case of the fast family of solitary waves.

As an example, graphs of the components u_1 and u_2 for solitary wave calculations in the isotropic case with rotation of the initial data, which have the form of (1.6), around the x axis, since, in these calculations there are two non-zero components of u and, in all the remaining cases, there is only one component. The propagation of an isotropic solitary wave is shown in Fig. 1 in the stable case (a) when $V = 0.8$ and the evolution of the initial data of the isotropic solitary wave type in the unstable case (b) when $V = 0.5$ and $\varphi = 0.3$ (the angle of rotation). The graphs of u_1 lie in the domain $u > 0$ and the graphs of u_2 lie in the domain $u < 0$. For clarity, the thickness of the lines for different instants of time has been made different. The common data for all the figures in this paper are: $f = 1$, $m = 1$, $\varkappa = 1$, $\rho_0 = 1$.

The qualitative form of the graphs is the same in all the calculations. In the stable cases, the solitary waves propagate without any change in form. In the unstable cases, the solitary wave initially moves without a change of form and then begins to self-focus which, in the final analysis, leads to an explosive

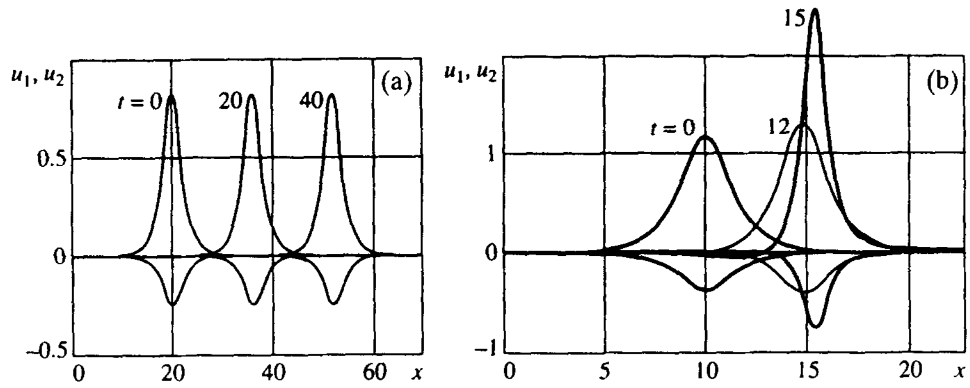

Fig. 1

growth in the amplitude and the halting of the calculation. The smaller the spatial step which is taken, that is, the more accurately the solitary wave is modelled, the later this occurs. In thie case, if, at the initial instant of time, one of the components is equal to zero, then it also remains equal to zero at subsequent instants of time. In the case of isotropic calculations with rotation, the relation $u_2/u_1 = -\text{tg}\varphi$, remains true for all instants of time up to the instant when the calculation stops.

The results of the calculations to check the stability of solitary waves belonging to the slow and fast families in the anisotropic case which have been presented above enable one to verify stability along just one direction. In these calculations, one of the components of the deformations remains equal to zero during the whole time of the calculation. However, the existence of an additional unstable direction in the case of the operator $\mathcal{H}$ in this case [3, 4] enables us to expect the fast solitary waves to be unstable over the whole range of values of V and not only when $V^2 < \mu_2/2$. With the aim of verifying this hypothesis, calculations with an initial orthogonal perturbation were carried out. The initial data for the component u_1 are now non-zero: $u_1 = \varepsilon u_2$, where ε is a small parameter. We note immediately that the instability being considered here is a slowly developing process and, therefore, in the segment $V^2 < \mu_2/2$, where explosive instability occurs, there is no sense in carrying out such calculations.

The calculations showed the following. When the propagation velocity of the wave V is close to $\mu_2^{1/2}$, the perturbation has practically no effect on the solitary waves (the slow component, which is progressively detached from the fast component, separates out). When the velocity is reduced and, correspondingly, there is an increase in the amplitude of the solitary waves, the periodic orthogonal component u_1 appears (the case of calculations, where the resonance wavelength is comparable with the characteristic wavelength of the solitary wave), see Fig. 2, $\varepsilon = 0.1$, $V = 1$, $g = 0.2$. Its spatial three-dimensionality is implied in the description of the process below and one bears in mind the fact that, in the case of the component u_2, the soliton component is mainly visible, while the periodic component u_1 is mainly visible in the case of u_1. To the right of the solitary wave (see Fig. 2), there is radiation, which is not in resonance with it and the phase velocity of the waves is greater than the velocity of the solitary wave. The spatial pattern outside the domain of the solitary wave has certain features in common with previously investigated cases of the quasistationary decay of a solitary wave when the line $\omega = Uk$ intersects the dispersion branch when $k \neq 0$ [10, 11]. Moreover, within the solitary wave domain (which is defined, for example, as a level of 1% of its amplitude), the solution for u_1 is found to be clearly an unsteady-state solution. In this domain, the form of the graph of the periodic, orthogonal component (u_1) is similar to a solitary wave with an envelope, a 1:1 soliton. Moreover, this solitary wave with an envelope is cyclically in phase and in anti-phase with the maximum of u_2. Oscillations with the maximum amplitude values of the envelope of the periodic component being studied occur simultaneously with these oscillations. The average steady-state values of the component (u_1) when $\varepsilon \to 0$ depend linearly on ε over a prolonged period of time. Correspondingly, the amplitude of the oscillations of the soliton component (the oscillations of the quantity $\max u_2(t)$, where the maximum is taken along the x coordinate) tends to zero when $\varepsilon \to 0$.

When long time intervals are being considered, the mean amplitude of the soliton component slowly decreases due to the existence of radiation, and the mean amplitude of the periodic orthogonal component in the domain within the solitary wave slowly increases; this occurs more slowly, the smaller the initial perturbation (weak instability).

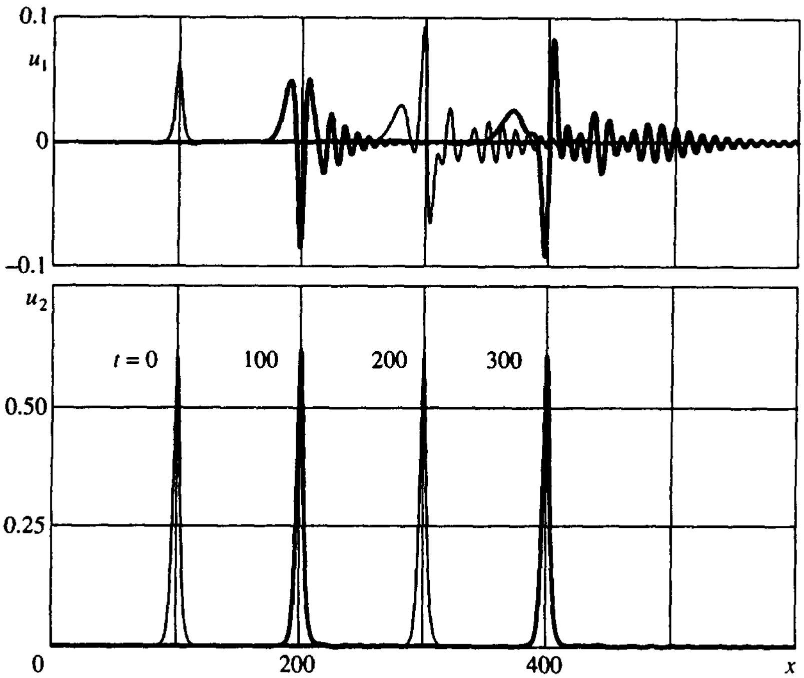

Fig. 2

In order to compare the qualitative properties of the process for different values of the velocity V and the amplitude of the solitary waves, the time-dependence of the maximum and minimum values of u_1 and u_2 is shown in Fig. 3 for initial data of the fast solitary wave type with an orthogonal perturbation at the fairly high value $V = 1$ (a) and at the lower value $V = 0.9$ (b). In both cases $V^2 > \mu_2/2$: 1 is $\max(u_2)$ (the amplitude of the soliton component), 2 is $\max(u_1)$ and 3 is $\min(u_1)$ (the quantities characterizing the oscillating amplitude of the periodic component). The graphs for $\varepsilon = 0, 1$ are shown by the solid curve in Fig. 3a and by the dashed curve for $\varepsilon = 0.025$. The solid and dashed curves in Fig. 3b correspond to $\varepsilon = 0.025$ and $\varepsilon = 0.0125$. The existence of discontinuities in the case of curves 2 and 3 is due to the fact that there is more than one local maximum or minimum in the solutions while, in the graphs, the absolute maximum and minimum values are shown which, at different instants of time, correspond to different local extrema.

For a smaller value of V, an initial perturbation, as small as desired, over a fairly short period of time leads to the appearance of an orthogonal, periodic component of finite amplitude and to a contraction of the amplitude of the solitary wave by a finite amount (strong instability). In this case, the wavelength of the solitary wave and its velocity increase, and so the resonance wavelength of the periodic component decreases. Subsequently, as in the case of a large value of V, an oscillatory process develops, which leads to a gradual further contraction of the mean amplitude of the solitary wave. Such behaviour always occurs if $V < \mu_1^{1/2}$ (when there is no resonance) but it also occurs for large values of V, when the wavelength of the solitary wave becomes approximately equal to the wavelength of the orthogonal component. Actually, in the case shown in Fig. 3b, the same processes which have been described above (an increase in the amplitude of the periodic component and a reduction in the amplitude of the solitary wave, see Fig. 3a) occur, but so rapidly that the dependence on the amplitude of the initial perturbation is lost shortly afterwards.

A control calculation to verify the stability of the slow solitary wave was also carried out for $V^2 > \mu_1/2$ with respect to the action of an orthogonal perturbation of a similar type. In this case, the results of the calculation were found to be in complete agreement with the theoretical results [3, 4] on the stability of a solitary wave.

3. DISCUSSION OF THE RESULTS

It follows from the results [3, 4] that:

– in an anisotropic material, slow solitary waves belonging to the slow family (1.5) are stable for velocities V in the range $I_{st} = \{V \in I, \mu_1/2 < V^2\}$;

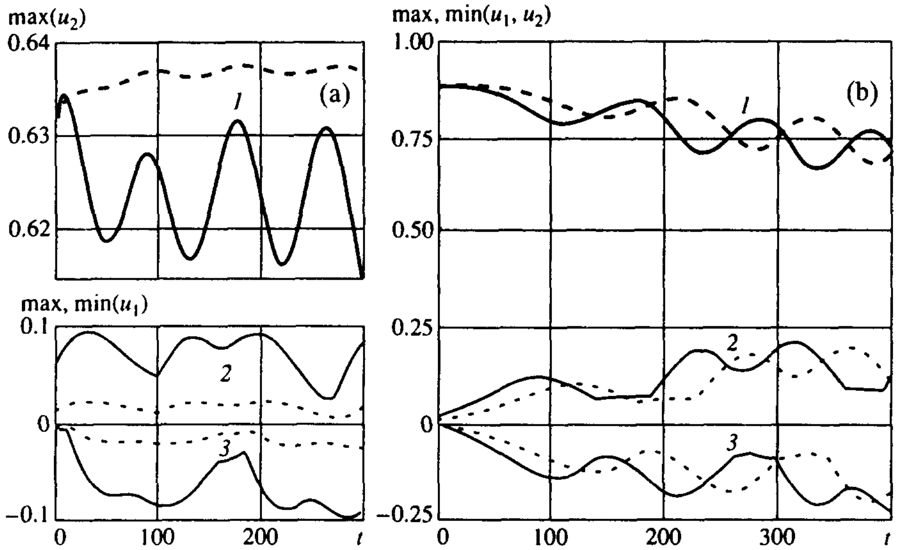

Fig. 3

– when there is no anisotropy, solitary waves belonging to family (1.6) are stable in the velocity range $I_{st} = \{V \in I, \mu/2 < V^2\}$.

Unlike long solitary waves on the surface of a heavy liquid in fluid dynamics, when the solitary waves are constructed in such a manner that an increase in amplitude occurs together with an increase in velocity, in the case of an elastic medium, we have a reduction in the amplitude when the velocity of the solitary wave increases. This is due to the different behaviour of the quantity Q, which is conserved as a result of translational invariance in the above model of fluid dynamics and the theory of elasticity; in the model of fluid dynamics it is an increasing function of the velocity while, in models of the theory of elasticity, it either decreases (for example, see [12]) or has a different behaviour for different velocity ranges (as in the case considered in this paper). The monotonicity of the function $Q(V)$ ensures the equivalence of the global and local stability of the soliton families of surface waves in fluid dynamics in the sense that, if the elements of a family from a small velocity range are stable, all the family is stable. In the case of an elastic medium considered in this paper, the situation is different: the stability of solitary waves in the above-mentioned velocity ranges (the stability range) does not imply the stability over the intervals additional to the stability range. Furthermore, as has been shown in this paper, solitary waves of the slow family in the anisotropic case when $V^2 < \mu_1/2$, of the fast family when $V^2 < \mu_2/2$, and non-anisotropic solitary waves when $V^2 < \mu/2$ will be unstable and, moreover, the instability is of an explosive character: the solitary waves collapse after a finite time, their amplitude tends to infinity and the carrier decreases. Explosive instability is associated with the following fact. Equations (1.1) without the higher derivatives, which have been linearized around the soliton solution, change their type from hyperbolic to elliptical in certain intervals of the x axis, beginning from a certain value of the velocity. We will now illustrate what has been said using the example of solitary waves belonging to the slow family in the anisotropic case, taking account of only the perturbation for the deformation component u_1. We now consider Eq. (1.1) for u_1, which has been linearized around the solitary wave, omitting the leading fourth-order derivative:

$$u_{1tt} = \left(\mu_1 - 3\frac{\varkappa}{\rho_0}u_1^{c2}\right)u_{1xx} - 3\frac{\varkappa}{\rho_0}(u_1^{c2})_{xx}u_1 - 6\frac{\varkappa}{\rho_0}(u_1^{c2})_x u_{1x} \tag{3.1}$$

The type of Eq. (3.1) being considered is defined by the sign of the coefficient of u_{1xx}. From (3.1) and (1.5), we have that this equation is hyperbolic everywhere with respect to x for $V \in I_h$

$$I_h = \left\{V \in I, V^2 > \frac{5}{6}\mu_1\right\} \tag{3.2}$$

and elliptic within a certain interval of the x axis outside I_h. It follows from (3.2) that $I_h \subset I_{st}$. In the "earlier stages of ellipticity" when V still belong to I_{st}, the interval of the x axis where Eq. (3.1) is elliptic, is sufficiently small and the instability which occurs on account of the ill-posed nature of the Cauchy problem for (3.1) is suppressed by the dispersion term with a fourth-order spatial derivative. If $V \in I$ leaves the stability range I_{st}, the x internal of ellipticity of (3.1) is found to be sufficiently large to affect the evolution of the Cauchy data having the form of a solitary wave. As a result, we have an unlimited increase in the amplitude of the solitary wave.

The operator $\mathcal{H}$ for the fast family of solitary waves in the anisotropic case has an additional unstable direction which corresponds to perturbations of the null component u_1^c of the fast family of solitary waves. When $V^2 > \mu_2/2$, these perturbations will destroy the stability of solitary waves belonging to the fast family. We will now evaluate the process of decay instability, which occurs in this case, from the point of view of wave theory. Suppose the wavelength of a solitary wave is much greater than the wavelength of the resonance orthogonal perturbation. It can be assumed that, for an investigation within the framework of geometrical optics, the soliton component plays the role of an extended perturbation which decreases the phase velocity of short orthogonal waves. In this connection, the solitary wave plays the role of a waveguide which captures part of the energy of the initial perturbation and there is a certain leakage of energy of the periodic wave due to the fact that the waveguide solution is asymptotically approximate and not exact. Some kind of non-linear mechanism exists for replenishing of the energy of the periodic component by the transfer of energy from the soliton component. It also leads to an oscillatory process which keeps the mean amplitude of the periodic component close to a constant value over a prolonged period of time. Obviously, the qualitative effects are also transferable to the case of a finite ratio of the wavelengths. In this case, the amplitude of the wave being studied must increase when the ratio of the wavelength of the solitary wave and the wavelength of the resonant radiation is reduced, which does occur in practice. However, if the wavelength of the periodic component becomes close to the characteristic wavelength of the solitary wave, it can be supposed that the capture of energy becomes impossible, as is also confirmed by calculations.

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